Mathematical Enrichment 5/3/16

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15 57 What amounts can we measure / obtain?

General problem Given positive integers m, n. Determine which numbers l can be obtained as a combination $s \cdot m + t \cdot n$ where s, t ~~on~~ are integers?

1) If d is a common divisor of m, n then d divides any combination $s \cdot m + t \cdot n$.
In particular, any 'obtainable' number must be a multiple of the $\gcd(m,n) = (m,n) = g$, say.

2) !!! g and any multiple of g IS obtainable (Euclid's algorithm).

Example 17, 57 So $g = 1$

$$57 = 3 \cdot 17 + 6$$
$$17 = 2 \cdot 6 + 5$$
$$6 = 5 + 1$$

$$\begin{aligned} 1 &= 6 - 5 \\ &= 6 - (17 - 2 \cdot 6) \\ &= 3 \cdot 6 - 17 \\ &= 3 \cdot (57 - 3 \cdot 17) - 17 \end{aligned}$$

$$\boxed{1 = 3 \cdot 57 - 10 \cdot 17}$$

$\boxed{437}$ $\boxed{986}$

$$986 = 2 \cdot 437 + 112$$
$$437 = 3 \cdot 112 + 101$$
$$112 = 101 + 11$$
$$101 = 9 \cdot 11 + 2$$
$$11 = 5 \cdot 2 + 1 = \text{gcd.}$$

$$1 = 11 - 5 \cdot 2 = 11 - 5 \cdot (101 - 9 \cdot 11) = 46 \cdot 11 - 5 \cdot 101$$
$$= 46 \cdot (112 - 101) - 5 \cdot 101 = 46 \cdot 112 - 51 \cdot 101$$
$$= 46 \cdot 112 - 51 \cdot (437 - 3 \cdot 112)$$
$$= 199 \cdot 112 - 51 \cdot 437$$
$$= 199 \cdot (986 - 2 \cdot 437) - 51 \cdot 437$$
$$\boxed{1 = 199 \cdot 986 - 449 \cdot 437}$$

Theorem $m, n \geq 1$

If $(m, n) = 1$ then the equation

$xm + yn = 1$ is solvable in integers;

There are integers s, t satisfying

$s \cdot m + t \cdot n = 1$ (and a method to find them).

(of course, converse is also true : (3)

if $1 = s \cdot m + n \cdot t$ for some s, t integers

then $(m, n) = 1$).

IMO Prove that the fraction

$$\frac{21n + 4}{14n + 3}$$

is irreducible for all $n \geq 1$.

i.e. show $(21n + 4, 14n + 3) = 1$

for all $n \geq 1$.

Solution: Observe that

$$3 \cdot (14n + 3) - 2 \cdot (21n + 4) = 1$$

for any n

$\Rightarrow$ any common divisor of $14n+3$, $21n+4$ also must divide 1.

Alternatively,

$$21n + 4 = 14n + 3 + 7n + 1$$

$$14n + 3 = 2 \cdot (7n + 1) + 1$$

$\boxed{3c}$ $\boxed{5c}$ stamps

What postage amounts can be obtained?

1	2	3	4	5	6	7	8	9	10	11	
×	×	✓	×	✓	✓	×	✓	✓	✓	✓	..

3 in a row (8, 9, 10)

If $l \geq 8$ ✓.

$\boxed{5c}$ $\boxed{8c}$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
×	×	×	×	✓	×	×	✓	×	✓	×	×	✓	×	✓	✓	×

18	19	20	21	22	23	24	25	26	27	28	29	30
✓	×	✓	✓	×	✓	✓	✓	✓	×	✓	✓	✓

31	32
✓	✓

If $l \geq \boxed{28}$ it is obtainable.

<u>The problem</u>: Given $m, n \geq 1$. Determine which integers $l \geq 1$ can be written

$$l = s \cdot m + t \cdot n$$

where s, t are <u>non-negative</u> integers (i.e. $s, t \geq 0$)

(Well suppose $(m,n) = 1$)

$5c \quad 8c \rightsquigarrow 19c$?

$$2 \cdot 8 - 3 \cdot 5 = 1$$

multiply by 19

$$38 \cdot \underline{8} - 57 \cdot \underline{5} = 19$$

1 step

$$-5 \cdot 8 + 8 \cdot 5$$

$$\underline{33} \cdot 8 - \underline{49} \cdot 5 = 19$$

6 steps

$3 \cdot 8 - 1 \cdot 5 = 19.$ no use.

<u>Theorem</u> Suppose $(m,n) = 1$ and suppose $m \mid nk$ then $m \mid k$

<u>Proof:</u> There exist s, t such that

$$1 = s \cdot m + t \cdot n$$

$$k = \underset{\substack{\uparrow \\ \text{multiple of } m}}{smk} + t \cdot \underset{\substack{\uparrow \\ \text{multiple of } m}}{nk} \Rightarrow k \text{ is a multiple of } m.$$

$(m,n) = 1$

Suppose $\quad sm + tn = \ell = s_1 m + t_1 n$

Then we must have

$$s_1 = s - an, \quad t_1 = t + am$$

for some integer a.

<u>Proof:</u> $\quad m\cdot(s - s_1) = n\cdot(t_1 - t)$

$$\Rightarrow \quad m \mid n\cdot(t_1 - t)$$

$$\underset{\text{Theorem}}{\Longrightarrow} \quad m \mid t_1 - t$$

i.e. $\quad t_1 - t = am \quad$ for some integer a

$$\Rightarrow \quad t_1 = t + am$$

But then

$$m\cdot(s - s_1) = n\cdot a\cdot m$$

$$\Rightarrow \quad s - s_1 = an = s_1 = s - an.$$

Back to <u>Stamp problem</u>

$(m,n) = 1 \qquad$ When can we solve $\ell = sm + tn \quad s,t \geq 0$
(Which ℓ's ?)

Certainly we can write

$$\ell = s\cdot m + t\cdot n$$

where $\quad 0 \leq s \leq n-1 \quad$ (adding or subtracting multiples of n where necessary).

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If $t \geq 0$, all is well.

Otherwise, $t \leq -1$.

In this case $l \leq (n-1)\cdot m + (-1)\cdot n$

$$l \leq nm - m - n$$

Thus if $\boxed{l > nm - m - n}$, $t \geq 0$

and l is obtainable.

(eg. $m=3, n=5$, $3\cdot 5 - 3 - 5 = 7$
$m=5, n=8$ $5\cdot 8 - 5 - 8 = 27$)

Is $nm - m - n$ obtainable?

No: Why? $\underbrace{nm - m - n}_{l} = \underbrace{(n-1)}_{s}\cdot m + \underbrace{(-1)}_{t}\cdot n$

only other sols $s = (n-1) - an < 0$
$t = -1 + am$ ← need $a > 0$

So $\boxed{nm - m - n}$ is never obtainable.

7c 11c

$7\cdot 11 - 7 - 11 = 59$ not obtainable.

But any amount ≥ 60 is obtainable.

Some problems (gcd's).

1. Suppose $sm - tn = \pm 1$

   Show $(m+t, n+s) = 1$.

2. Find $g = \gcd(2^8+1, 2^{32}+1)$

   Express g as $s \cdot (2^8+1) + t \cdot (2^{32}+1)$

   [Use algebra.]

3. Suppose $\gcd(m,n) = 1$

   Show $\gcd(m^2-n^2, 2mn) = 1$ or 2

4. $m, n \geqslant 1$ $\gcd(m,n) = d$.

   $a > 1$

   Show $\gcd(a^m-1, a^n-1) = a^d-1$.